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Critical Nonlinearities in Partial Differential Equations

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"...it is ingrained in mathematical science that every real advance goes hand in hand with the invention of sharper tools and simpler methods which at the same time assist in understanding earlier theories and cast aside older more complicated developments."
David Hilbert, Mathematical Problems, 1900

Abstract. This paper focuses on the role of critical nonlinearities within the framework of global solvability of nonlinear PDE's. In particular, we present some new approach to blow-up issues for nonlinear problems.

1. Introduction

Critical nonlinearities are essentially divided into three classes:

(I) Fujita type critical nonlinearities;
(II) Bernstein type critical nonlinearities;
(III) Sobolev type critical nonlinearities.

and all of them are related to *the Hamlet problem:*

"to be or not to be global solutions of nonlinear PDEs".

In this paper we mainly consider Fujita type critical nonlinearities, whereas for Bernstein type critical nonlinearities we refer the interested reader to [20] – [23], where the general functional approach is developed for finding Bernstein type critical exponents in Sobolev spaces. About Sobolev

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type critical nonlinearities we refer to [18], [19], and [8], where the method of variational identities is exploited to obtain Sobolev type critical exponents.

The main purpose of this paper is to present in a highlights fashion some new approach in determining Fujita type critical exponents in different nonlinear PDE's; for this purpose, in what follows we adopt an examples-layered presentation.

In order to show the main differences between local and global properties of solutions to nonlinear PDE's, let us begin with two model examples.

Example 1.1. Consider a parabolic equation

$$\begin{cases} u_t = \Delta u + u^p \text{ in } \mathbb{R}^+ \times \mathbb{R}^N \quad (p > 1), \\ u|_{t=0} = u_0(x) \geq 0, \quad u_0(x) \not\equiv 0. \end{cases} \tag{1.1}$$

Local analysis of this problem shows that for each $p > 1$ and initial function u_0 there exists a $T = T(u_0)$ such that at least a local (in time) solution $u(t, x)$ exists for $0 < t < T$ and all $x \in \mathbb{R}^N$.

The aim of global analysis is to determine whether $T = \infty$ may occur.

In the celebrated paper by H. Fujita [2] it was shown that there exists a critical exponent $p_{\mathrm{f}} = 1 + \frac{2}{N}$ such that for $1 < p < p_{\mathrm{f}}$ and $u_0 \geq 0$ no global solution of (1.1) exists. Namely, for each $u_0 \geq 0$ there exists $T_\infty = T_\infty(u_0) < \infty$ such that for any solution $u(t, x)$ to (1.1) one has $\int |u|^p(t, x)\, dx \to +\infty$, as $t \to T_\infty$.

Example 1.2. Now consider the following hyperbolic equation

$$\begin{cases} u_{tt} = \Delta u + |u|^p \text{ in } \mathbb{R}^+ \times \mathbb{R}^N \quad (p > 1), \\ u|_{t=0} = u_0(x), \\ u_t|_{t=0} = u_1(x) \geq 0 \ (u_1(x) \not\equiv 0). \end{cases} \tag{1.2}$$

Here local analysis shows that for each $p > 1$, u_0 and u_1 one can find $T = T(u_0, u_1)$ such that there exists a local (in time) solution to (1.2) $u(t, x)$ for $0 < t < T$ and $x \in \mathbb{R}^N$.

Global analysis of (1.2) was carried out by T. Kato in [9]. He obtained a critical exponent $p_{\mathrm{k}} = \frac{N+1}{N-1}$ such that for $1 < p \leq p_{\mathrm{k}}$ there is no global in t (i.e., for all $t > 0$) solution to (1.2), that is, for each u_0, u_1 there exists $T_\infty = T_\infty(u_0, u_1) < \infty$ such that $\int |u|^p(t, x)\, dx \to +\infty$, as $t \to T_\infty$.

All the above results were obtained by comparison techniques which are based on the following:

1. comparison (maximum) principle, positivity of the fundamental solution of the corresponding linear operator;
2. self-similar analysis of solutions.

This method actually reduces the class of nonlinear equations which can be handled, to scalar second-order ones. However, by this approach a wide class of quasilinear partial differential equations has been investigated, see for instance [3], [4], [5], [11], [12], [1], and [31].

On the other hand, it is clear that this classical approach fails in general in the case of higher-order equations.

In the joint work [15] with E. Mitidieri, we suggested a new approach towards blow-up problems which relies on the notion of Nonlinear Capacity.

2. The nonlinear capacity method: a general scheme

It turns out that the blow-up of solutions is connected with the "nonlinear capacity" induced by nonlinear operators. Before stating rigorous definitions, which will be given in the next section, let us first give a flavor of the key ingredients involved by considering some model examples in which one can see how the nonlinear capacity "appears" and "works".

Example 2.1. Consider as a pilot example the following,

$$\begin{cases} -\Delta u \geq u^q \text{ in } \mathbb{R}^N, \\ u \geq 0, \quad q > 1. \end{cases}$$

related to the nonlinear operator $A(u) := -\Delta u - u^q$. By solution to this problem we mean a function $u \in L^q_{\mathrm{loc}}(\mathbb{R}^N)$, such that

$$\int u^q \psi \leq -\int u \Delta\psi \ \text{ for all } \psi \in C_0^2(\mathbb{R}^N), \psi \geq 0.$$

Let $e = e_R = \{x \in \mathbb{R}^N \ : \ |x| \leq R\}$, $\psi \in C_0^2(\mathbb{R}^N)$, $\psi \geq 0$ and

$$\psi(x) = \begin{cases} 1, & |x| \leq R, \\ 0, & |x| \geq \kappa R \ (\kappa > 1). \end{cases}$$

We have

$$\int u^q\psi \leq -\int \Delta u \cdot \psi = -\int u\Delta\psi \leq \left(\int u^q\psi\right)^{1/q} \left(\int \frac{|\Delta\psi|^{q'}}{\psi^{q'-1}}\right)^{1/q'}$$

and thus

$$\int\limits_{|x|\leq R} u^q \leq \int u^q\psi \leq \int \frac{|\Delta\psi|^{q'}}{\psi^{q'-1}}. \tag{2.1}$$

We define the nonlinear capacity as

$$\mathrm{Cap}(A, e_R) := \inf_{\psi} \int \frac{|\Delta\psi|^{q'}}{\psi^{q'-1}}$$

and hence by replacing the right-hand side in (2.1) with $\mathrm{Cap}(A, e_R)$ one sees how *the nonlinear capacity appears as the optimal a priori estimate within a certain class of test functions.*

Next we show how the nonlinear capacity "works".

Clearly, if $\mathrm{Cap}(A, e_R) \to 0$, as $R \to \infty$, there is no nontrivial (entire) solution to our problem. We have

$$\mathrm{Cap}(A, e_R) \le \int \frac{|\Delta\psi|^{q'}}{\psi^{q'-1}}$$

and after scaling $\psi(x) = \psi_0\left(\frac{|x|}{R}\right), \quad \psi_0 \ge 0, \quad \psi_0 \in C^2,$

$$\text{with } \psi_0(\rho) = \begin{cases} 1, & \rho \le 1, \\ 0, & \rho \ge \kappa > 1, \end{cases}$$

we get

$$\int \frac{|\Delta\psi|^{q'}}{\psi^{q'-1}} = R^{N-2q'} \int \frac{|\Delta\psi_0|^{q'}}{\psi_0^{q'-1}}.$$

Hence if $1 < q < q_{\mathrm{cr}} = \begin{cases} \frac{N}{N-2} & \text{for } N > 2, \\ +\infty & \text{for } N = 1, 2, \end{cases}$ then

$$\mathrm{Cap}(A, e_R) \to 0, \text{ as } R \to \infty \text{ and in turn } \int_{\mathbb{R}^N} u^q = 0,\ u \ge 0$$

which yields $u = 0$ a.e. in $\mathbb{R}^N$.

In the limit case $N = 2q'$ we have $\mathrm{Cap}(A, e_R) \le c_* < \infty$ for all $R > 1$, which implies that $\int\limits_{\mathbb{R}^N} u^q < \infty$ and then

$$\begin{aligned}
\int\limits_{|x|\le R} u^q \le \int u^q\psi \le -\int u\Delta\psi = -\int\limits_{\mathrm{supp}\Delta\psi} u\Delta\psi \\
\le \Big(\int\limits_{\mathrm{supp}\Delta\psi} u^q\psi\Big)^{1/q} \Big(\int \frac{|\Delta\psi|^{q'}}{\psi^{q'-1}}\Big)^{1/q'} \\
\le c_*^{1/q'} \cdot \Big(\int\limits_{R\le|x|\le\kappa R} u^q\psi\Big)^{1/q} \to 0 \text{ as } R \to \infty.
\end{aligned}$$

It is known (see for instance [15]) that the critical exponent $q_{\rm cr}$ is sharp. Indeed, this follows from the following counterexample: if $N > 2$, for each $q > q_{\rm cr}$ a function $u(x) = A(1+|x|^2)^\lambda > 0$ with $\dfrac{2-N}{2} < \lambda < -\dfrac{1}{q-1}$ and $A = 2|\lambda| \cdot (N-2+2\lambda) > 0$ is a solution of the inequality $-\Delta u \geq u^q$ in the whole space $\mathbb{R}^N$.

Example 2.2. In a similar way one can manage the non-homogeneous version of the Fujita problem:

$$\begin{cases} u_t = \Delta u + |u|^p + f(t,x) \text{ in } \mathbb{R}^+ \times \mathbb{R}^N \\ u|_{t=0} = u_0(x). \end{cases}$$

Clearly the critical exponent depends now on f and u_0, that is, $p_{\rm cr} = p_{\rm cr}(N, f, u_0)$.

Let

$$\int\limits_{|x|\leq R} \int\limits_0^T f(t,x)\,dt\,dx\Big|_{T=R^2} + \int\limits_{|x|\leq R} u_0(x)\,dx \geq c_d(1+R^\gamma)$$

with $\gamma \geq 0$ and $c_d > 0$ (without the assumptions $f \geq 0$, $u_0 \geq 0$!). Then

$$p_{\rm cr} = \begin{cases} +\infty, & \text{if } \gamma \geq N, \\ 1 + \frac{2}{N-\gamma}, & \text{if } 0 \leq \gamma < N. \end{cases}$$

Moreover, one obtains the following estimate for the domain of existence of the solution:

$$\begin{aligned} &T_\infty < T_* = (c_*/c_d)^\theta, \quad \theta = \frac{2(p-1)}{(\gamma-N)p+N+2-\gamma} \text{ for } 1 < p < p_{\rm cr}, \\ &R_\infty < R_* = T_*^{1/2}, \end{aligned}$$

where the exponent θ turns out to be sharp.

The main advantage of this approach consists in the *homotopic stability* of critical exponents. Indeed, let $A_0(u)$ be a nonlinear operator with critical exponent q_0 and let $A_1(u)$ be another nonlinear operator. Moreover, let $A(t,u)$ be a one parameter family of nonlinear operators, $t \in [0,1]$ and such that $A(0,u) = A_0(u)$ and $A(1,u) = A_1(u)$. Let also the coefficients in $A(t,u)$ be not degenerate, i.e., uniformly bounded away from zero and infinity (in the suitable norm) for any $t \in [0,1]$ and for any admissible u. Then the critical exponent q_1 of A_1 coincides with q_0.

As a concrete example consider the following:

Example 2.3. Let $q > 1$ and A be the operator defined by

$$A(u) : = -\Delta u - u^q, \quad u \geq 0, \quad \text{in } \mathbb{R}^N.$$

As we have shown, the corresponding critical exponent is

$$q_{\text{cr}}(A) = \begin{cases} \frac{N}{N-2} & \text{for } N > 2, \\ +\infty & \text{for } N \leq 2. \end{cases}$$

Now let

$$\tilde{A}(u) = -\text{div}(a(x, u, Du) \cdot Du) - b(x, u, Du), \; u \geq 0$$

with Carathéodory functions a, b such that

$$\begin{cases} 0 < a_0 \leq a(x, u, Du) \leq a_1, \\ b(x, u, Du) \geq b_0 |u|^q, \; b_0 > 0. \end{cases} \tag{H}$$

Then one has $q_{\text{cr}}(\tilde{A}) = q_{\text{cr}}(A)$.

In particular, under the above assumptions, for the operator $\tilde{A}_1$:

$$\tilde{A}_1(u) = -\text{div}\left(a(x, u, Du)\frac{Du}{\sqrt{1 + |Du|^2}}\right) - b(x, u, Du)$$

we have

$$q_{\text{cr}}(\tilde{A}_1) = q_{\text{cr}}(\tilde{A}) = q_{\text{cr}}(A)$$

and clearly the same holds for

$$A_1(u) = -\text{div}\frac{Du}{\sqrt{1 + |Du|^2}} - u^q, \; u \geq 0.$$

We conclude this section by resuming a general scheme:

$$\boxed{\textit{nonlinear operator}} \Rightarrow \boxed{\textit{nonlinear capacity}}$$
$$\Rightarrow \boxed{\textit{capacity dimension}} \Rightarrow \boxed{\textit{blow} - \textit{up conditions}}$$

The main advantages of the method are:

- generality and flexibility,
- simplicity,
- sharpness (non-improvability) of the obtained criteria,
- homotopic stability of critical exponents with respect to nonlinear perturbations.

Clearly, the optimality of the obtained results is related to the framework under considerations; for instance, it is well known from Example 2.1 that the derived critical exponent is sharp as far as we are concerned with the *inequality* in the *whole space*.

3. The notion of Nonlinear Capacity

Next we recall from [24] the definition of nonlinear capacity induced by nonlinear operators.

Let $X_{\rm loc}(\Omega) \subset L_{1,{\rm loc}}(\Omega)$ be a function space ($\Omega \subset \mathbb{R}^N$ is a domain), and let A be an operator $A\colon X_{\rm loc}(\Omega) \to D'(\Omega)$, where $D'(\Omega)$ denotes the space of distributions. Let $e \subset \Omega$ be compact and

$$C_0^\infty(e,\Omega) := \{\psi \in C_0^\infty(\Omega) \,|\, 0 \le \psi \le 1 \text{ in } \Omega \text{ and } \psi = 1 \text{ in } e\}.$$

Let us introduce the following quantity:

$$\mathrm{Cap}_A(e,\Omega) = \inf_\psi \sup_u \{(A(u),\psi) \,\big|\, u \in X_{\rm loc}(\Omega),\ \psi \in C_0^\infty(e,\Omega)\} \tag{3.1}$$

where $(\cdot,\cdot)$ denotes a "scalar product".

Definition 3.1. We call the quantity $\mathrm{Cap}_A(e,\Omega)$ defined by (3.1) nonlinear capacity.

Remark 3.2. In the case A is a coercive operator, we also require in the above definition:

$$\|\psi\|_X \le \|u\|_X.$$

Example 3.3. Let $A(u) = -\Delta u$ and $X_{\rm loc}(\Omega) = W^{1,2}_{\rm loc}(\Omega)$. In this case

$$\begin{aligned}\mathrm{Cap}_A(e,\Omega) &= \inf_\psi \sup_u \left(-\int_\Omega \Delta u \cdot \psi \, dx \right) \\ &\qquad \text{where } u \in W^{1,2}_{\rm loc}(\Omega),\ \|u\|_{1,2} \le \|\psi\|_{1,2},\ \psi \in C_0^\infty(e,\Omega) \\ &= \inf_{\psi \in C_0^\infty(e,\Omega)} \int_\Omega |\nabla \psi|^2 \, dx.\end{aligned}$$

Thus $\mathrm{Cap}_\Delta(e,\Omega)$ turns out to be the classical (harmonic) capacity.

Example 3.4. Let $A(u) = -\Delta_p u$, where $\Delta_p u := \mathrm{div}(|Du|^{p-2}Du)$ with $p > 1$, and $X_{\rm loc}(\Omega) = W^{1,p}_{\rm loc}(\Omega)$. Then

$$\mathrm{Cap}_A(e,\Omega) = \inf_\psi \left\{ \int_\Omega |\nabla \psi|^p \, dx,\ \psi \in C_0^\infty(e,\Omega) \right\}.$$

Then $\mathrm{Cap}_{\Delta_p}(e,\Omega)$ is the so-called p–harmonic capacity; see for instance [14].

In order to cover a wide class of nonlinear operators, we next generalize the above definition.

It is well-known (see for instance [13]) that the general theory for nonlinear equations is based on a priori estimates for solutions. A priori estimates, in turn, are mainly derived by using special multipliers. In this respect, we introduce the following definition for pair (A, M) consisting of an operator A and a multiplier M:

Definition 3.5. Let A_M be determined by

$$A_M = MA$$

such that $MA\colon X_{\mathrm{loc}}(\Omega) \to D'(\Omega)$. Then

$$\begin{aligned} \mathrm{Cap}_{A_M}(e,\Omega)\colon &= \mathrm{Cap}_{MA}(e,\Omega) \\ &= \inf_{\psi} \sup_{u} \int\limits_{\Omega} MA(u)\psi\,dx,\ u \in X_{\mathrm{loc}}(\Omega)\ \psi \in C_0^\infty(e,\Omega). \end{aligned} \tag{3.2}$$

Example 3.6. (Nonlinear elliptic capacity)
Let $A(u)\colon = -\Delta_p u - u^q,\ p > 1,\ q > p-1$ and

$$X_{\mathrm{loc}}(\Omega) = \left\{ u \in W^{1,p}_{\mathrm{loc}}(\Omega) \,\middle|\, u \geq 0, \int\limits_{\mathrm{loc}} |Du|^p u^\alpha + \int\limits_{\mathrm{loc}} u^{q+\alpha} < \infty \right\}.$$

Take $M(u) = u^\alpha$ with $1 - p < \alpha < 0$. Then

$$\mathrm{Cap}_{A_M}(e,\Omega) = \inf_{\psi \in C_0^\infty(e,\Omega)} \left\{ \frac{1}{c} \int\limits_{\Omega} \frac{|D\psi|^\gamma}{\psi^{\gamma-1}} dx \right\}$$

with some $c > 0$, where $\gamma = \dfrac{p(q+\alpha)}{q-p+1}$.

Remark 3.7. If we take $\psi = \zeta^\gamma$, then

$$\mathrm{Cap}_{A_M}(e,\Omega) = \inf_{\psi \in C_0^\infty(e,\Omega)} \left\{ \frac{\gamma^\gamma}{c} \int\limits_{\Omega} |D\zeta|^\gamma dx \right\}.$$

From here it follows (see [15]) that

$$q_{\mathrm{cr}} = \begin{cases} \dfrac{N(p-1)}{N-p} & \text{if } N > p, \\ +\infty & \text{if } p \geq N. \end{cases}$$

In particular, for $p = 2$ we get the exponent $q_{\mathrm{cr}} = \dfrac{N}{N-2}$ $(N > 2)$.

Example 3.8. (Nonlinear parabolic capacity)
Let

$$A(u):\ = \frac{\partial u}{\partial t} - D(u^\sigma |Du|^{p-2} Du) - u^q,\ u \geq 0 \text{ in } \mathbb{R}_+^{N+1},$$

where we set $\mathbb{R}_+^{N+1} := \mathbb{R}^+ \times \mathbb{R}^N$ and where $p > 1$, $\frac{p}{N} + \sigma + p - 1 > 1$, $q > \max\{1, \sigma + p - 1\}$. Then

$$\begin{aligned}&\mathrm{Cap}_{A_M}(e, Q)\\ &= \inf_{\psi} \left\{ \iint\limits_{Q} \frac{|D\psi|^{\gamma_1}}{\psi^{\gamma_1 - 1}} + \iint\limits_{Q} \frac{|\psi_t|^{\gamma_2}}{\psi^{\gamma_2 - 1}} : \psi = \psi(t,x),\ \psi \in C_0^\infty(e, Q),\ Q \subset \mathbb{R}_+^{N+1} \right\}\end{aligned}$$

with

$$\gamma_1 = \frac{p(q+\alpha)}{q - (\sigma + p - 1)},\ \gamma_2 = \frac{q+\alpha}{q-1} \quad (-1 < \alpha \leq 0).$$

From here we get (see [15])

$$q_{\mathrm{cr}} = \frac{p}{N} + \sigma + p - 1.$$

This critical exponent was previously obtained by V.A. Galaktionov (see [3], [4]).

Related to the homotopic stability of the critical exponent, we have the following: let $\tilde{A}$ be a parabolic operator with variable coefficients

$$\tilde{A}(u):\ = \frac{\partial u}{\partial t} - \mathrm{div}(a(\dots) u^\sigma |Du|^{p-2} Du) - b(\dots), \quad u \geq 0 \text{ in } \mathbb{R}_+^{N+1}$$

with

$$\begin{aligned} a(\dots) &= a(t, x, u, Du, \dots),\\ b(\dots) &= b(t, x, u, Du, \dots)\end{aligned}$$

under the assumptions:

$$\begin{cases} 0 < a_0 \leq a(\dots) \leq a_1, \\ b(x, u, Du) \geq b_0 |u|^q,\ b_0 > 0. \end{cases}$$

Of course, we assume that these nonlinear coefficients are Carathéodory functions for which the operator $\tilde{A}$ turns out to be well defined in the related function space. Then we have

$$q_{\mathrm{cr}}(\tilde{A}) = q_{\mathrm{cr}}(A) = \frac{p}{N} + \sigma + p - 1.$$

This technique extends to higher-order equations as we show in the next

Example 3.9. (The generalized Zeldovich-Kompaneets-Barenblatt equation) This equation is determined by the operator A_0,

$$A_0(u)\colon = u_t + (-\Delta)^k|u|^m - |u|^p \quad \text{in} \quad \mathbb{R}^{N+1}_+$$

with $p > m \geq 1$. Then we have

$$\operatorname{Cap}_{A_0}(e,\Omega) = \inf_{\psi\in C_0^\infty(e,\Omega)} \left\{ \iint\limits_{\Omega} \frac{|D^k\psi|^{q'}}{\psi^{q'-1}} + \frac{|\psi_t|^{p'}}{\psi^{p'-1}} \right\}$$

with $q = \dfrac{p}{m}$, $q' = \dfrac{q}{q-1}$, $p' = \dfrac{p}{p-1}$ and the critical exponent is given by

$$p_{\text{cr}}(A_0) = m + \frac{2k}{N}.$$

Let us determine the dependence of the critical exponent of this operator on the data of the Cauchy problem. To this end, we consider the following inequality:

$$\begin{cases} A_0(u) \geq f(t,x) & \text{in } \mathbb{R}^{N+1}_+, \\ u|_{t=0} = u_0(x) & \text{in } \mathbb{R}^N \end{cases}$$

with $\displaystyle\int\limits_0^T \int\limits_{B_R} f(t,x)\,dt\,dx\Big|_{T=R^\theta} + \int\limits_{B_R} u_0(x)\,dx \geq c_d(1+R^\gamma)$, $\theta = 2k\dfrac{p-1}{p-m}$ and $c_d > 0$.

Then the critical exponent p_{cr} of this problem is equal to

$$p_{\text{cr}} = \begin{cases} m + \frac{2k}{p-\gamma} & \text{for } 0 \leq \gamma < N, \\ +\infty & \text{for } \gamma \geq N. \end{cases}$$

Note that the critical exponent is homotopically stable in this case as well. Indeed, consider

$$\tilde{A}(u) = u_t + (-\Delta)^k f(t,x,u) - b(t,x,u),$$

with Carathéodory functions f and b satisfying the inequalities

$$|f(t,x,u)| \leq c|u|^m,$$
$$b(t,x,u) \geq b_0|u|^p$$

with $p > m > 1$, $b_0 > 0$. Then we have

$$p_{\text{cr}}(\tilde{A}) = p_{\text{cr}}(A_0).$$

Example 3.10. (Nonlinear hyperbolic capacity)

Let

$$A(u) := \frac{\partial^2 u}{\partial t^2} - \sum_{l \leq |\alpha| \leq L} D^\alpha A_\alpha(t,x,u) - b(t,x,u)|u|^q \text{ in } \mathbb{R}_+^{N+1}$$

with Carathéodory functions A_α and b such that

$$\begin{aligned} |A_\alpha(t,x,u)| &\leq c|u|^p, \quad p > 0, \\ b(t,x,u) &\geq b_0 > 0 \end{aligned}$$

with $q > \max\{1, \sigma + p - 1\}$. Then we have

$$\mathrm{Cap}_A(e,\Omega) \leq C \inf \iint_\Omega \frac{|D\psi|^{\gamma_1}}{\psi^{\gamma_1 - 1}} + \iint_\Omega \frac{|\psi_{tt}|^{\gamma_2}}{\psi^{\gamma_2 - 1}}$$

where the infimum is taken over the functions $\psi = \psi(t,x)$, $\psi \in C_0^\infty(e,\Omega)$, $\Omega \subset \mathbb{R}_+^{N+1}$ and where

$$\gamma_1 = \frac{q}{q-p}, \ \gamma_2 = \frac{q}{q-1},$$

from which we get (see [15])

$$q_{\mathrm{cr}} = \begin{cases} \dfrac{2N+l}{2N-l} & \text{for } 2N - l > 0, \\ +\infty & \text{for } 2N - l \leq 0. \end{cases}$$

Remark 3.11. For $A(u) := \dfrac{\partial^2 u}{\partial t^2} - \Delta u - |u|^q$ in $\mathbb{R}_+^{N+1}$ we have $p = 1$, $l = 2$ and thus

$$q_{\mathrm{cr}}|_{p=1,l=2} = \frac{2N \cdot 1 + 2}{2N \cdot 1 - 2} = \frac{N+1}{N-1} \ (N > 1),$$

that is the Kato exponent [9].

Further details, developments and applications of the nonlinear capacity method can be found in [15], [16], [25].

4. Nonlocal nonlinearities

The technique we have explained so far can be used also to deal with nonlinear integral equations and inequalities. In this context, we recall from [17] some Liouville type theorems which can be obtained by this method.

Example 4.1. Consider the equation

$$u(x) = \int\limits_{\mathbb{R}^N} \frac{1}{|x-y|^{N-\beta}} |u(y)|^q dy. \tag{4.1}$$

For $0 < \beta < N$ which is the integral version of $-\Delta_\beta u = |u|^q$ in $\mathbb{R}^N$. Then we have $q_{\text{cr}} = \dfrac{N}{N-\beta}$.

Theorem 4.2. *Let* $1 < q \le q_{\text{cr}}$. *Then* (4.1) *has no nontrivial solutions.*

Example 4.3. Consider a more general case

$$L(u) \ge \int\limits_{\mathbb{R}^N} \frac{1}{|x-y|^{N-\beta}} |u(y)|^q dy \tag{4.2}$$

with $\beta < N$, where L is a linear differential operator.

Theorem 4.4. *There exists* $q_{\text{cr}} > 1$ *such that for* $1 < q \le q_{\text{cr}}$ *equation* (4.2) *has no nontrivial solution.*

Example 4.5. For the parabolic nonlocal inequality

$$u_t - \Delta u \ge \int\limits_{\mathbb{R}^N} \frac{1}{|x-y|^{N-\beta}} |u(y)|^q dy,$$

we have

$$q_{\text{cr}} = \frac{N+2}{N-\beta} \quad (0 < \beta < N).$$

Example 4.6. For the hyperbolic nonlocal inequality

$$u_{tt} - \Delta u \ge \int\limits_{\mathbb{R}^N} \frac{1}{|x-y|^{N-\beta}} |u(y)|^q dy,$$

we obtain

$$q_{\text{cr}} = \begin{cases} \dfrac{N+1}{N-1-\beta} & \text{for } 0 < \beta < N-1, \\ +\infty & \text{for } N-1 \le \beta < N. \end{cases}$$

Example 4.7. We now consider the following higher-order Cauchy problem:

$$\begin{cases} \dfrac{\partial^k u}{\partial t^k} - \displaystyle\sum_{l \le |\alpha| \le m} D^\alpha a_\alpha(t,x,u) \ge \int\limits_{\mathbb{R}^N} |x-y|^{\beta-N} |u(t,y)|^q \, dy + f(t,x), \\ \dfrac{\partial^i u}{\partial t^i}\Big|_{t=0} = u_i(x), \quad i = 0, \dots, k-1. \end{cases} \tag{4.3}$$

Theorem 4.8. *Let $k \geq 1$, $l \geq 1$, $N > \beta > 0$ and functions $u_i \in L^1_{\text{loc}}(\mathbb{R}^N)$ and $f \in L^1_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^N)$ be such that*

$$\lim_{R\to\infty}\inf \int\limits_{B_R} u_{k-1}(x)\,dx \geq 0, \quad \lim_{R,T\to\infty}\inf \int\limits_0^T \int\limits_{B_R} f(t,x)\,dx\,dt \geq 0.$$

Let a_α satisfy the estimate $|a_\alpha(t,x,u)| \leq c_\alpha |u|^p$ with some $c_\alpha > 0$, $p > 0$ and with $q > \max\{1,p\}$. Then problem (4.3) has no nontrivial solution for

$$q \leq q_{\text{cr}} = \begin{cases} \dfrac{(kN-\beta)p + l + \beta}{(N-\beta-l)k + l} & \text{if } (N-\beta-l)k + l > 0, \\ +\infty & \text{if } (N-\beta-l)k + l \leq 0. \end{cases}$$

Corollary 4.9. *Consider the generalized John problem*

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u \geq \displaystyle\int\limits_{\mathbb{R}^N} |x-y|^{\beta-N} |u(t,y)|^q\,dy + f(t,x) \text{ in } \mathbb{R}^{N+1}_+, \\ u|_{t=0} = u_0(x) \text{ in } \mathbb{R}^N, \\ \dfrac{\partial u}{\partial t}\Big|_{t=0} = u_1(x) \text{ in } \mathbb{R}^N, \end{cases} \tag{4.4}$$

where $q > 1$ and $N > \beta > 0$. Consider functions $u_0, u_1 \in L^1_{\text{loc}}(\mathbb{R}^N)$, $f \in L^1_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^N)$,

$$\lim_{R\to\infty}\inf \int\limits_{B_R} u_1(x)\,dx \geq 0, \quad \lim_{R,T\to\infty}\inf \int\limits_0^T \int\limits_{B_R} f(t,x)\,dx\,dt \geq 0.$$

Then problem (4.4) has no nontrivial solution for

$$q \leq q_{\text{cr}} = \begin{cases} \dfrac{N+1}{N-1-\beta} & \text{if } 0 < \beta < N-1, \\ +\infty & \text{if } N-1 \leq \beta < N. \end{cases}$$

The proof of the above statements is based on the nonlocal (integral) capacity generated by the corresponding nonlocal (integral) operator; see [17].

5. The Kuramoto-Sivashinsky Equation

Let us consider the following equation:

$$u_t + \Delta^2 u + \Delta u = |Du|^2, \quad \text{in } \mathbb{R}^N_+ \tag{5.1}$$

which was introduced by Y. Kuramoto [10] to describe dissipative structures in connection with diffusion and independently by G. Sivashinsky [30] in the context of hydrodynamical instability in combustion theory; later on, other physical applications were discovered. So far an extensive literature has been devoted to this equation, though the available results mainly deal with the one-dimensional case; here we present some results in the multi-dimensional case.

Let Ω_0 be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega_0$ and $0 \in \Omega_0$, $\Omega = \mathbb{R}^N \setminus \overline{\Omega}_0$ and let $Q_T = (0,T) \times \Omega$ and $\Gamma_T = (0,T) \times \partial\Omega_0$. Consider the problem

$$u_t + \Delta^2 u + \Delta u = |Du|^2 \text{ in } Q_T \tag{5.2}$$

with boundary conditions

$$u = \frac{\partial}{\partial n}(u + \Delta u) = 0 \text{ on } \Gamma_T \tag{5.3}$$

and initial datum

$$u|_{t=0} = u_0(x) \text{ in } \Omega. \tag{5.4}$$

The problem is considered in the space $W^{1,4}_{2,\mathrm{loc}}(Q_T)$.

By using the approach based on the nonlinear capacity method we obtain the following blow-up result; see [28], [7] and [29].

Theorem 5.1. *Let*

$$\overline{\lim_{R\to\infty}} \frac{I_R(u_0)}{R^{\frac{3N-2}{2}}} = +\infty \qquad \textit{for } N > 2,$$

$$\overline{\lim_{R\to\infty}} \frac{I_R(u_0)}{R^2 \ln R} = +\infty \qquad \textit{for } N = 2,$$

$$\overline{\lim_{R\to\infty}} \frac{I_R(u_0)}{R} = +\infty \qquad \textit{for } N = 1,$$

where $I_R(u_0) := \int\limits_{\Omega_R} u_0(x)\,dx$ *and* $\Omega_R := \{x \in \mathbb{R}^N \mid \ |x| \le R\}$*. Then problem* (5.2)–(5.4) *has no global solution for all* $t > 0$*. Moreover, there is no solution* $u(t,x)$ *for* $x \in \Omega_R$ *with* $R > R_*$ *and* $t > T_{R_*}$*, where* R_* *and* T_{R_*} *are determined by the initial data of the problem.*

Remark 5.2. It should be noted that the asymptotic exponent of the behavior of the initial data u_0, namely $\dfrac{3N-2}{2}$ for $N > 2$, is sharp (non-improvable). This follows from a counterexample which shows the existence of a global solution to the problem when

$$|u_0(x)| \le C|x|^{\frac{N-2}{2}} \quad (N > 2,\ R > 1),$$

which implies

$$\left| \int\limits_{\Omega_R} u_0(x)\,dx \right| \le CR^{\frac{3N-2}{2}} \text{ for } N > 2,\ R > 1.$$

6. Schemes for applying nonlinear capacity

I. The explicit scheme.
Consider the inequality $A(u) \ge f(u) \ge 0$ in $X_{\mathrm{loc}}(\mathbb{R}^N)$, where the function f is such that $f(u) = 0$ implies $u = 0$ a.e. in $\mathbb{R}^N$.

Fix $B_R \subset \mathbb{R}^N$ and $\psi \ge 0$, $\psi \in C_0^\infty(\mathbb{R}^N)$, $\psi(x) = 1$ for $x \in B_R$. Then we have

$$\int\limits_{B_R} f(u) \le \int f(u)\psi \le \int A(u)\psi \le \mathrm{Cap}_A(B_R, \mathbb{R}^N)$$

by the *very* definition.

Hence, if $\mathrm{Cap}_A(B_R, \mathbb{R}^N) \to 0$ as $R \to \infty$, we get $u = 0$ a.e. in $\mathbb{R}^N$.

The "size" of the possible existence domain can be also estimated according to the following scheme.

Let $A(u) \ge h \ge 0$ in $X_{\mathrm{loc}}(\mathbb{R}^N)$. Then we have

$$\int h\psi \le \int A(u)\psi \le \mathrm{Cap}_A(B_R, \mathbb{R}^N) \text{ for } R \ge 1.$$

If $\int\limits_{B_R} h \ge c_1 \cdot R^\kappa$ for $R \ge 1$ and $\mathrm{Cap}_A(B_R, \mathbb{R}^N) \le c_2 \cdot R^\theta$ for $R \ge 1$, we get

$$c_1 R^\kappa \le c_2 R^\theta, \quad \kappa > \theta.$$

Thus the radius of the existence domain can be estimated by

$$R_* = \max\left\{1, (c_2/c_1)^{1/(\kappa-\theta)}\right\}.$$

II. The implicit scheme.
Consider a non-negative functional of the form

$$\int\limits_{\mathbb{R}^N} E(u)\,dx, \quad u \in X^+_{\mathrm{loc}}(\mathbb{R}^N)$$

such that $E(u) \geq 0$ and $\int\limits_{\Omega} E(u)\,dx = 0$ implies $u = 0$ a.e. in Ω.

If for any solution $u \in X^{+}_{\mathrm{loc}}(\mathbb{R}^N)$ of the inequality $A(u) \geq 0$ there holds

$$\int\limits_{\Omega} E(u)\psi \leq c_0 \mathrm{Cap}_A(e_R, \mathbb{R}^N).$$

Then one has $u = 0$ a.e. in $\mathbb{R}^N$, provided $\mathrm{Cap}_A(e_R, \mathbb{R}^N) \to 0$, as $R \to \infty$.

7. Some generalizations: the k-th order entropy and related applications

The above applications of the nonlinear capacity method rely on the positivity of the right-hand side of the nonlinear equation. Otherwise, we cannot apply this approach directly. For instance, we cannot apply this approach directly to conservation laws. In order to overcome this difficulty and extending the setting in which this approach can be exploited, we introduce the notion of k-th order entropy.

Suppose that for a given non-stationary problem in $\mathbb{R}^{N+1}_{+}$ there holds an inequality of the form

$$\frac{\partial H_0}{\partial t} - \sum_{i=1}^{N} \frac{\partial H_i}{\partial x_i} \geq W \text{ in } \mathbb{R}^{N+1}_{+} \tag{7.1}$$

Here $H_0, H_1, \ldots, H_N$ and W are functions depending on t, x, the solution u of the problem and its derivatives of order $k \geq 0$ that belong to the class C^1 in the range of the corresponding variables.

Let these functions satisfy the inequality

$$W \geq c_0(|H_0|^q + \sum_{i=1}^{N} |H_i|^{q_i}) \tag{7.2}$$

for some constants $c_0 > 0$, q, $q_1, \ldots, q_N > 1$ in the whole range of the arguments under consideration.

Definition 7.1. A pair of functions $(H_0, \overline{H})$, where H_0 is a C^1 scalar function and $\overline{H} = (H_1, \ldots, H_N)$ is a C^1 vector valued function satisfying inequalities (7.1)–(7.2) on the solutions of the problem, is called the k-th order entropy for the original non-stationary problem.

We remark that the class of solutions turns out to be determined by the conditions

$$\begin{cases} H_0,\, H_i,\, W \in L^1_{\mathrm{loc}}(\mathbb{R}^{N+1}_+), \\ H_0|_{t=0} \in L^1_{\mathrm{loc}}(\mathbb{R}^N). \end{cases} \tag{7.3}$$

Definition 7.2. A generalized solution of the non-stationary problem satisfying conditions (7.3) is called a k-entropy one, if for this solution there exists the k-th order entropy satisfying inequality (7.1) in the sense of distributions $D'_+(\mathbb{R}^{N+1}_+)$.

A key feature in this context being the following quantity,

$$\theta = \sum_{i=1}^{N} \frac{q'}{q'_i} + 1 - q' \text{ with } q' = \frac{q}{q-1} \text{ and } q'_i = \frac{q_i}{q_i - 1} \ (i = 1, \ldots, N), \tag{7.4}$$

which arises in applications to the existence of global solutions to non-stationary problems.

Theorem 7.3. *Let the non-stationary problem under consideration have the k-th order entropy with $\theta \le 0$. Then this non-stationary problem does not admit a nontrivial global k-entropy solution in the whole $\mathbb{R}^{N+1}_+$ if the initial conditions are such that*

$$H_0|_{u(0,x)} \ge 0. \tag{7.5}$$

The proof of this theorem and applications to conservation laws can be found in [26].

Example 7.4. (The Hamilton–Jacobi equation). Following [27], we present as an application of the previous theorem, the Hamilton–Jacobi equation with a non-negative Hamiltonian:

$$\begin{cases} \dfrac{\partial u}{\partial t} = H(u, Du), & (x,t) \in \mathbb{R}^2_+, \\ u(0,x) = u_0(x) \ge 1, & x \in \mathbb{R}. \end{cases} \tag{7.6}$$

Let us define for this problem the 1st order entropy as the vector valued function (H_0, H_1) with $H_0 = H_0(u,p)$, $H_1 = H_1(u,p)$ satisfying (7.1)–(7.2) on smooth solutions of the equation (7.6).

Then validity of inequality (7.1) for all values of the corresponding arguments implies by (7.2)

$$\frac{\partial H_1}{\partial p} = \frac{\partial H}{\partial p} \cdot \frac{\partial H_0}{\partial p}. \tag{7.7}$$

By choosing, for instance,

$$H_0(u,p) = u^a |p|^b, \qquad u \ge 1,\ p \in \mathbb{R},$$

we get

$$\frac{\partial H_1}{\partial p} = bu^a|p|^{b-2}p\frac{\partial H}{\partial p}.$$

Hence for

$$W(u,p) := \frac{\partial H_0}{\partial t} - \frac{\partial H_1}{\partial x}$$

with

$$\frac{\partial u}{\partial t} = H(u,p),\ p = \frac{\partial u}{\partial x}$$

we find

$$W(u,p) = a(1-b)u^{a-1}|p|^b H(u,p) \\ + b(b-1)u^{a-1}p\int (aH(u,s) + uH_u(u,s))|s|^{b-2}ds.$$

As a consequence of Theorem 7.3. we have the following

Corollary 7.5. *Let a non-negative C^1 Hamiltonian H be such that there exist parameters a and $b \in \mathbb{R}$ and functions H_0, H_1, W as above for which there holds*

$$W \geq c_0(|H_0|^q + |H_1|^{q_1})$$

for all $u \geq 1$ and $p \in \mathbb{R}$ with some $c_0 > 0$, $q > 1$ and $q_1 > 1$ such that

$$\theta = \frac{q'}{q_1'} + 1 - q' \leq 0.$$

Then there is no nontrivial global solution of the Cauchy problem (7.6) *with C^1-initial condition $u_0(x) \geq 1$.*

Example 7.6. Consider the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = u^\mu |Du|^\lambda, & (t,x) \in \mathbb{R}^2_+, \\ u(0,x) = u_0(x) \geq 1, & x \in \mathbb{R} \end{cases} \tag{7.8}$$

with $u_0 \in C^1(\mathbb{R})$. We restrict ourselves to the case $\lambda > 1$, $\lambda + \mu > 1$. Then for parameters a and b such that $b > 1$ and $a + b + \lambda + \mu - 1 \leq 0$ all the assumptions of Corollary 7.5. are fulfilled with

$$q = \frac{b+\lambda}{b},\ q_1 = \frac{b+\lambda}{b+\lambda-1} \text{ and } \theta = \frac{1-b}{\lambda} < 0.$$

Hence there is no nontrivial entropy solution of the Cauchy problem (7.8) for all $t > 0$. In particular, this problem has no nontrivial (i.e. $u \not\equiv$const) C^2_x solution for all $t > 0$.

Remark 7.7. It is worth to notice that the solutions under consideration, in general do not belong to the class of viscosity solutions.

8. Further applications

A mathematical model for a roof crash. Let us assume that a roof can be modeled in terms of a nonlinear membrane:

$$\frac{\partial^2 u}{\partial t^2} - \mathrm{div}\frac{Du}{\sqrt{1+|Du|^2}} = -k|u|^{q-1}u + h(x) \;\text{ in } \mathbb{R}^{N+1}_+,$$

where:

- u is a displacement of the roof from equilibrium,
- $k > 0$ is the rigidity coefficient,
- $-k|u|^{q-1}u$ is the restoring nonlinear force with $q > 1$,
- $h(x)$ is the external load.

Then the steady state of the roof $u \le 0$ (the u–axis is directed downwards) is governed by the equation

$$-\mathrm{div}\frac{Du}{\sqrt{1+|Du|^2}} = k|u|^q + h(x),\;\; h(x) \equiv H_0 \text{ in } \mathbb{R}^N.$$

By means of the approach discussed so far, for $h(x) \equiv H_0 > 0$ there exists R_∞ such that there is no solution in the ball B_R with $R \ge R_\infty$.

For R_∞ one also has the following bound

$$R_\infty \le R_* = \left(\frac{c_*}{H_0}\right)^{\frac{1}{2q'}},$$

where $q' = \dfrac{q}{q-1}$, $c_* = c_*(N,k)$.

Nonlinear capacity and existence. Actually the notion of nonlinear capacity can be somehow exploited also for existence problems, as we are going to show in the next example.

Example 8.1. Consider as a reference model the following Cauchy problem

$$\begin{cases} \dfrac{du}{dt} = u^2, \\ u|_{t=0} = u_0. \end{cases} \tag{8.1}$$

The explicit solution of this problem has the form $u(t) = \dfrac{u_0}{1-u_0 t}$. Hence for $u_0 < 0$ there exists a global solution, and for $u_0 > 0$, there is a blow-up solution with $T_\infty = \dfrac{1}{u_0}$.

Now we apply the nonlinear capacity approach to this problem. We have

$$\int_0^T u^2\varphi = u\varphi\Big|_0^T - \int_0^T u\varphi',$$

where

$$\varphi(t) \geq 0, \quad \varphi \in C_0^1(\mathbb{R}),$$
$$\varphi(T) = 0, \quad \varphi(0) = 1.$$

This yields

$$\int_0^T u^2\varphi \leq -u_0 + \left(\int_0^T u^2\varphi\right)^{1/2}\left(\int_0^T \frac{\varphi'^2}{\varphi}\right)^{1/2}. \tag{8.2}$$

Due to scaling

$$\varphi(t) = \varphi_0(\tau), \quad \tau = t/T,$$
$$\varphi_0 \in S := \{\varphi_0 \geq 0,\ \varphi_0(0) = 1,\ \varphi_0(1) = 0,\ \varphi_0 \in C_0^1(\mathbb{R})\} \tag{S}$$

we obtain

$$\int_0^T \frac{\varphi'^2}{\varphi}\, dt = \frac{1}{T}\int_0^1 \frac{{\varphi_0'}^2}{\varphi_0}\, d\tau.$$

This equality and (8.2) *give*

$$\int_0^T u^2\varphi \leq -u_0 + \frac{C_{\varphi_0}^{1/2}}{\sqrt{T}}\left(\int_0^T u^2\varphi\right)^{1/2} \tag{8.3}$$

with

$$C_{\varphi_0} = \int_0^1 \frac{{\varphi_0'}^2}{\varphi_0}\, d\tau.$$

Consider

$$\min\left\{\int_0^1 \frac{{\varphi_0'}^2}{\varphi_0}\, d\tau \colon \varphi_0 \in S\right\}.$$

After a change of variables

$$\varphi_0(\tau) \to \psi_0(\tau): \quad \varphi_0 = \psi_0^2$$

we get

$$\int_0^1 \frac{\varphi_0'^2}{\varphi_0}\, d\tau = 4 \int_0^1 \psi_0'^2\, d\tau.$$

Then

$$\min \left\{ \int_0^1 \psi_0'^2 d\tau : \psi_0 \in S \right\} = 1.$$

Thus $C_{\varphi_0} = 4$ and (8.3) leads to

$$\int_0^T u^2\varphi \le -u_0 + \frac{2}{\sqrt{T}} \left(\int_0^T u^2\varphi \right)^{1/2} \quad \Rightarrow$$

$$\left[\left(\int_0^T u^2\varphi \right)^{1/2} - \frac{1}{\sqrt{T}} \right]^2 \le -u_0 + \frac{1}{T}.$$

Consequently:

- If $u_0 < 0$, then $-u_0 + 1/T > 0$, hence due to the a priori estimate for ODE there exists a solution for any $T > 0$.
- If $u_0 > 0$ and $T < 1/u_0$, then $-u_0 + 1/T > 0$, and there exists a solution for $T < 1/u_0$.
- If $u_0 > 0$ and $T \ge 1/u_0$, we have $-u_0 + 1/T \le 0$ and there is no solution for $T \ge 1/u_0$.

Thus the blow-up time

$$T_\infty = 1/u_0$$

is the same as the one obtained before from the explicit form of the solution!

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